Stretching a triangle to infinity

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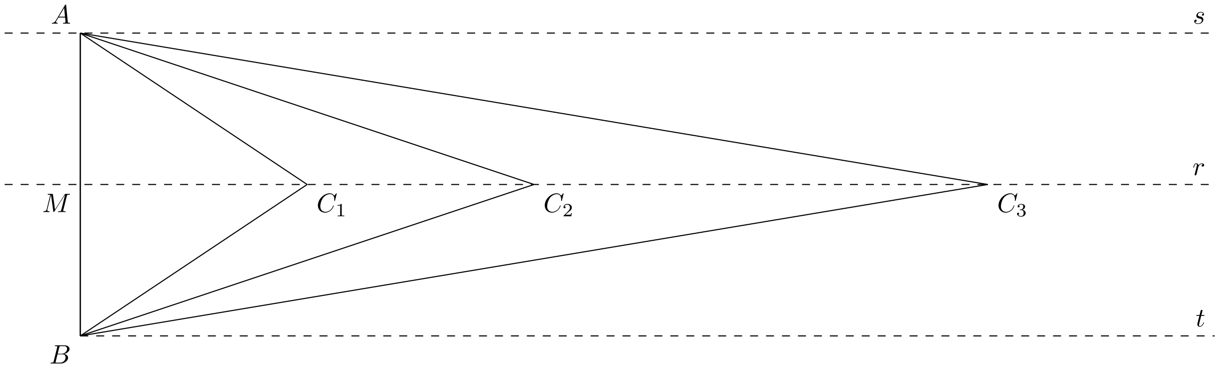
Question 1. Consider a triangle $\mathcal{T} = ABC$ and let a line r intersect the vertex C and the side AB. Imagine now to move C along r. What happens to angle $\widehat{ACB}$?



It is easy to see (and prove) that $\widehat{AC_1B} > \widehat{AC_2B} > \widehat{AC_3B}$ and that in general $\widehat{ACB}$ decreases as the distance between C and AB increases. Without loss of generality, assume r to be both the perpendicular and the median to AB as in the picture above. It follows that $tan(\frac{1}{2}\widehat{ACB}) = \frac{AM}{MC}$, and so

$$\widehat{ACB} = 2 \cdot tan^{-1}(\frac{AM}{MC}).$$

Therefore, $\widehat{ACB} \to 0$ as C gets further from AB.

Question 2. What would be the "most natural" point at infinity of angle $\widehat{ACB}$ when triangle $\mathcal{T}$ is subject to the former process of infinite stretch? More formally, consider the infinite sequence $C_1, C_2, \ldots, C_n, \ldots$ and the associated sequences $\widehat{AC_1B}, \widehat{AC_2B}, \ldots, \widehat{AC_1B}, \ldots$. If we *assume* the existence of the point at infinity C_∞ what would then be a most natural choice for $\widehat{AC_\infty B}$?

Since, as we have seen above, $\lim_{n\to\infty} \widehat{AC_nB} = 0$, it seems that $\widehat{AC_\infty B} = 0$ would be a natural choice. However, this line of reasoning, based on a reflection of a purely symbolic nature[1] is not satisfying. Let us consider the infinite sequence $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n, \ldots$, then what about T_∞?

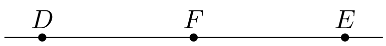
[1] i.e. based on a Fregean...

1) On the one hand, setting $\widehat{AC_\infty B} = 0$ would imply $AB = 0$ (at least this is true for all finite triangles). But this is impossible, since AB remains constant.

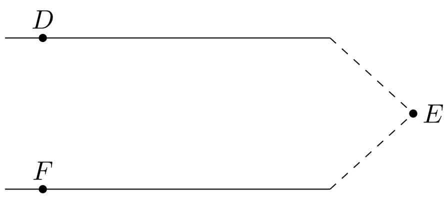
2) More importantly, as C_n moves further and further from AB, angles $\widehat{C_nAB}$ and $\widehat{C_nBA}$ tend to get closer and closer to be right, i.e. as n grows, the two sides AC_n and BC_n tend to be more and more parallel to each other, so to speak. Thus, at infinity, two lines should not intersect anymore.

All this arguably means that it is not possible to continuously extend the figure of a triangle to infinity without breaking the triangle itself. Therefore, in topological terms, the natural "point at infinity" of a triangle ABC subject to an infinite stretch would not be a triangle anymore, but two parallel lines!

Is this perhaps one of those few cases in which symbolic reasoning has the collateral effect of leading us astray badly? Indeed, it seems that, by considering the trigonometric formula above, the natural extension of the sequence of more and more stretched triangles would be an "infinite triangle" ABC with angle $\widehat{ACB} = 0$. However, since after the procedure of infinite stretch AC_∞ and BC_∞ are infinite (on the right), the two sides will never actually meet in C_∞, because such point is nowhere to be found on the plane. On the other hand, in the Question above we have assumed the existence of this point at infinity[2], so that we could just extend in a clever way the definition of "triangle" as to include ABC_∞. Furthermore, we could extend the definition of "angle of measure zero" as to apply (by decree) to the angle centered at points at infinity, so that $\widehat{DEF} = 0$ would imply



or



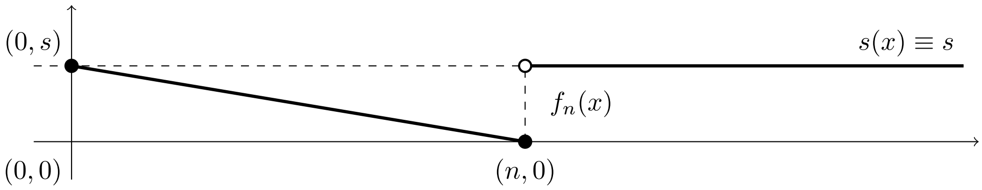
depending on whether E is a regular point or a point at infinity. In this way the formalism is accommodated as to match our geometrical intuition about the infinite sequence of triangles.

Question 3. Imagine a machine endowed with vision and capacity to conjure up formalisms for describing its own visual experience. How would such a machine be made to discover this simple accommodation of formalism?

Finally, as a side mathematical question, we may ask ourselves, looking at the first picture,

Question 4. How exactly do AC and BC tend to coincide with lines s and t respectively as MC grows larger? From the point of view of mathematical analysis, it is easy to show that this is a case of pointwise convergence and definitely not a case of full functional convergence.

[2]by extending the Euclidean plane in the same way as is common practice in projective geometry.



$$f_n(x) = \begin{cases} s - x/n & \text{if } 0 \leq x \leq n \\ s & \text{if } x > n \end{cases}$$

How does the sequence of functions $f_n(x)$ converges to the constant function $s(x) \equiv s$? It is easy to see that the following is not true:

$$\forall \epsilon > 0, \exists n_\epsilon, \forall n > n_\epsilon, \forall x (s - f_{n_\epsilon}(x) < \epsilon).$$

Howeverer, it is true that

$$\forall x, \forall \epsilon > 0, \exists n_\epsilon, \forall n > n_\epsilon, (s - f_{n_\epsilon}(x) < \epsilon).$$

Using mathematical analysis' jargon, then, we can say that in this is a case of pointwise convergence and definitely not a case of full functional convergence.