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VISUALIZATION AS VISION, IMAGINATION AND INTUITION: REFLECTIONS ON GRADUATE STUDENTS STRUGGLING WITH A VISUAL CONJECTURING PROBLEM

Francesco Beccuti
Università di Torino, Italy

In this paper we present some considerations on a case-study involving 25 graduate students in mathematics who were asked to conjecture the solution to a mathematical visualization problem. Surprisingly, no one of the subjects identified the correct solution and defended instead an incorrect one. We interpret the results within a fresh theoretical framework which classifies visualization skills into visual, imaginative and intuitive skills. We argue that the difficulties the students experienced may be ascribed to the considerable amount of imaginative effort needed in order to solve this particular problem (failure in imagination) as well as to the subjects' tendency to overgeneralize (false deduction derived from failure in intuition) together with (at least for a portion of the students) a difficulty in thinking out of the particular didactic contract they assumed to be in place. Finally, we conclude with a reflection on the institutional setting the subjects were immersed into and suggest directions for further investigations by linking psycho-pedagogical considerations on visualization with related university curriculum reform.

INTRODUCTION, PRELIMINARY DEFINITIONS AND RESEARCH QUESTION

Research in visualization within mathematics education originated in the work of Alan Bishop and was later carried out by various authors: see (Presmeg 2014) for a compendium and see also the report (Kadunz & Yerushalmy 2015) for a survey on recent advances in visualization. In this paper, we will follow the mainstream lineage of research developed by Abraham Arcavi and Norma Presmeg albeit explicitly stressing on some hopefully clarifying preliminary definitions inspired from the work of psychologist Efraim Fischbein as well as from the writings of mathematicians Felix Klein, David Hilbert and Henry Poincaré. These novel definitions, while allowing us to enclose the theoretical discourse concerning visualization within finer lines of demarcation, will also furnish us with the right vocabulary for framing the discussion of the case-study presented below.

Vision may be defined unambiguously as the faculty by which we directly see things which are there for us to see. On the other hand, *imagination* is the faculty by which we see what is not there to see (in mathematics this usually happens in connection with some properties one wants to prove or show). It may be divided into passive imagination (the act of representing to the mind something prompted to us from an outside source) and active imagination (the act of representing to the mind something not prompted from the outside). Furthermore, *intuition* is the faculty by which we generalize the properties that we see or imagine.

Finally, in accordance with the definition given in (Arcavi 2003) we define *visualization* in mathematics as all that concerns the faculties/properties/abilities above.

Visualization: the mode by which we bring mathematical objects at the attention of our senses, we manipulate them and we reflect on them, internally (i.e. in the mind) or externally via some material support, traditionally by hand-drawing on paper or, nowadays, by means of software-generated images.

Notice that the definition of intuition given above (essentially Fischbein's) is somewhat more specific than the usual meaning given to the term "intuition" (mostly found within philosophy of mathematics or mathematicians' introspective accounts) which is generally an umbrella term used by authors to characterize any informal way of grasping mathematical truths outside of formal reasoning. Indeed, for the great majority of authors "intuitive reasoning" is nothing but a synonym of "informal reasoning". Notice also that for simplicity and adherence to tradition, we take in this paper a clear a priori distinction between informal and formal reasoning, albeit agreeing with the philosophical stance taken in (Giardino 2010) that the two forms of reasoning are really inextricably intertwined. Notice also that the literature has traditionally distinguished between internal and external act of visualization. In (Presmeg 2006) this distinction is assumed as unproblematic by adopting the Piagetian view that any act of external visualization depends on internal mental images. We do not delve into this issue here, but we remark that the distinction must be made at the level of imagination, i.e. the distinction does not concern vision (always external) and intuition (always internal).

To get a concrete grasp of these definitions, let us look at the usual proof of following proposition: the opposite sides of a parallelogram are congruent to each other. If we draw two pairs of parallel lines as in Figure 1.a, we can *see* the parallelogram $ABCD$ as a direct act of *vision* (provided that we indeed know what a parallelogram is). Furthermore, in order to prove the proposition, we may *imagine* the segment AC (Figure 1.b) and consider the angles this new segment forms with the lines. We are then able to conclude that angles DAC and ACB are congruent to each other (Figure 1.c) as well as angles CAB and ACD (Figure 1.d). Therefore, the opposite sides of the initial parallelogram are congruent to each other. Finally, it is by *intuition* that we realize that the property thus proved is not linked to the particular parallelogram considered, but holds in general for all parallelograms.

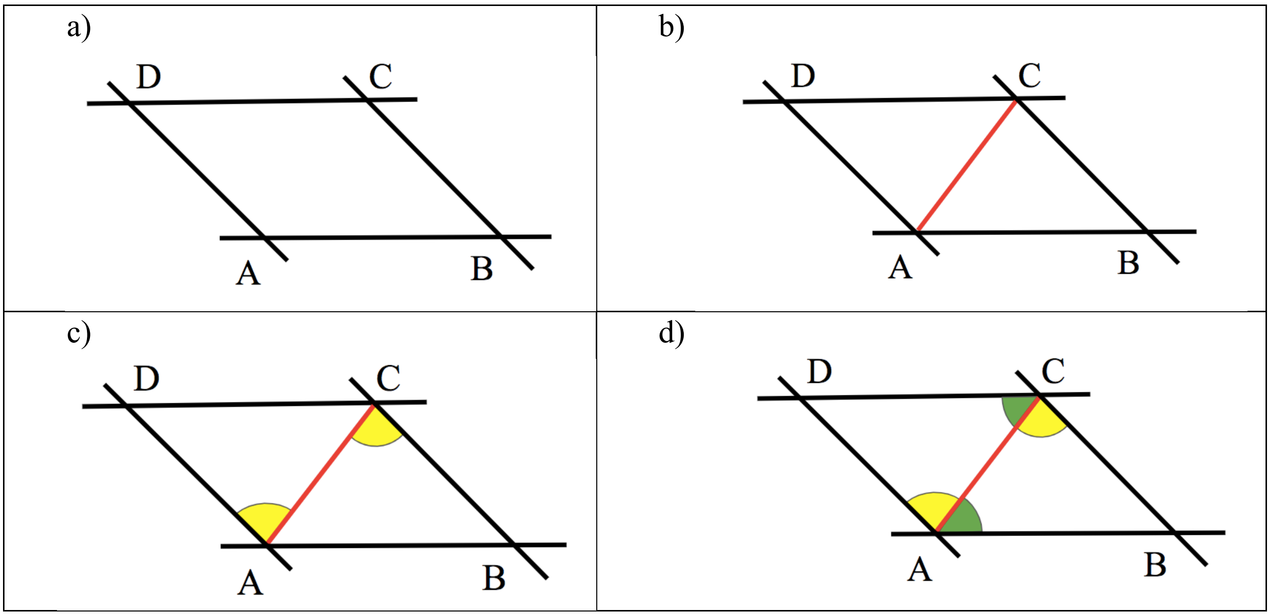

Figure 1.

Now, in the passage above the crucial imaginative step (the proverbial "idea" one must have) is to consider the segment AC and thus reduce the proof of the proposition to the proof of the congruence of triangles DAC and ACB. In this and in similar examples, the segment or shape that we need to imagine in order to complete the proof is almost evident to the trained eyes of a mathematician. However, this may not be the case for inexperienced pupils and, similarly, even experienced mathematicians might have trouble when the necessary imaginative step is not cued by the given figure or diagram.

Indeed, this is precisely the situation unfolding in the case-study presented below, the discussion of which will then allow us to pose ourselves the following research question:

How do we account for mathematically literate students' surprising failure in solving a visual conjecturing problem?

We will try to answer to this question by employing a qualitative analysis (centred on the threefold definition of visualization just given) of students' written accounts of their handling of what has proven to be a particularly challenging word-problem.

GRADUATE STUDENTS TRYING TO SOLVE A VISUALIZATION PROBLEM

The problem below was given to 28 first-year students following their first course in mathematics education while being enrolled in a master in mathematics of an Italian university which attracts many students graduated from other Italian universities. Indeed, the main requirement for entering the master is to have completed a three-year bachelor in pure or applied mathematics with good marks. Such bachelors, in the Italian university system (which does not offer a major-minor arrangement of credits but focuses instead almost entirely on mathematics only) invariably revolve around traditionally presented mathematical content knowledge in the form of theorems and proofs together with problem sets usually revolving on the application of these.

Problem: Four cities are placed at the four corners of a square and an engineer wants to design a road which connects them. What path she has to choose in order to use the least amount of materials?

The problem is equivalent to the problem of finding the minimal path which connects the vertices of a square. The optimal solution to the problem is presented in red in Figure 2.d (modulo a 90-degree rotation), while Figure 2.a, 2.b and 2.c show paths which indeed connect the four roads but are not minimal.

Furthermore, the following hint was given right after the statement of the problem: "the solution is not the path consisting of the square itself" in order to help the students exclude right away the path presented in Figure 2.a. However, this may have prompted instead some confusion since a portion of students interpreted it "normatively", so to speak, as we will see below.

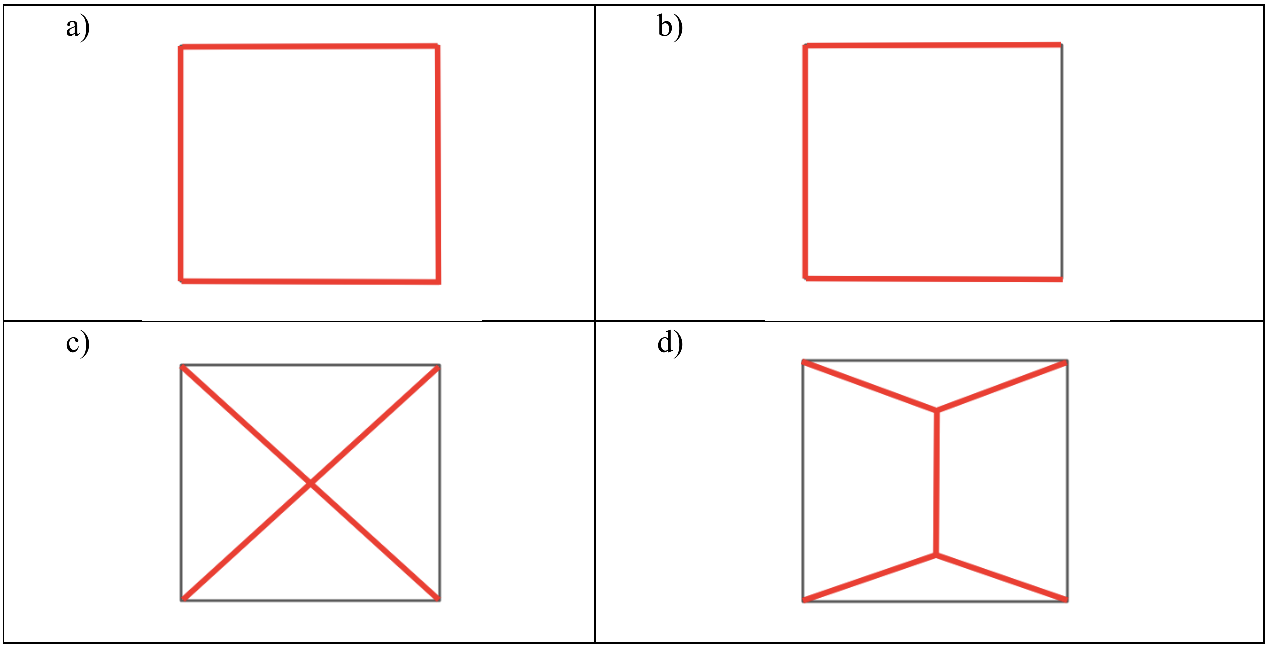

Figure 2.

The participants were asked to form groups of three or four. In each group one student (the observer) had to write a report observing and elaborating on the way she and her colleagues arrived at the solution. In the spirit of (Arzarello et al. 2002), the students were invited to visualize the problem by means of a dynamic-geometry software (Geogebra) and given about 45 minutes to solve the problem collaboratively. Surprisingly, only one of the eight groups hinted at the correct solution. However, the students in this group admitted that one of them had already seen the problem before and hence they were excluded from the study. Among the other groups, four suggested that the solution was the one depicted in Figure 2.c: "the diagonals", while three groups were convinced that the solution of (their interpretation of) the problem was the one depicted in Figure 2.a, "the square", and thus had to conceptually "accommodate" the aforementioned hint, as we will discuss.

Analysis of two reports of students who concluded that the solution is the diagonals

Let us now examine the reports of two representative groups (here called A and B) of the former portion of students and interpret them within our theoretical framework. The remaining groups (C and D) had similar reports. Indeed, Group A drew a square together with its diagonals, and just wrote the following laconic sentence.

> The minimal path to unite the 4 cities is through the bisectors of the quadrilateral, given the fact that a straight segment is always the shortest way to unite 2 points

It seems that the procedure these students went through is that of a false deduction, or an over-generalization described in (Fischbein 1987): since the shortest way to unite two points is a straight line, then the shortest way to unite four points is simply two straight lines. However, the report does not furnish us with any clue as to how they arrived to consider the path consisting of the diagonals, or "the bisectors", as they say here. On the other hand, a quotation from Group B's report may let us understand how indeed these other students arrived at the same false conjecture: the observer writes that his colleagues

> […] decided to represent the diagonals of the quadrilateral, since they thought that the best idea was that of starting from the properties offered by the quadrilateral […] they [then] asked themselves if there did not exist a path better than the one just deduced […] they then decided to construct a second quadrilateral and conjoin the vertices, not by the diagonals, but by segments located in a different way […] In conclusion both the girls agreed, in light of their reasoning and the tests performed, that the minimal path was the one represented by the diagonals of the quadrilateral.

Indeed, the students in Group B chose the diagonals because they were "offered by the quadrilateral" itself. In other words, the decision to conjoin opposite points in the parallelogram example above was, as they seem to mean, suggested or cued by the figure itself.

Furthermore, these students also went through a process of overgeneralization (or failure in intuition), as they considered two different configurations (their own drawings in the software are displayed in Figure 4.a and 4.b) and noticed that the path consisting of the diagonals was shorter than those, thus (wrongly) concluding that the former is shorter than any possible path.

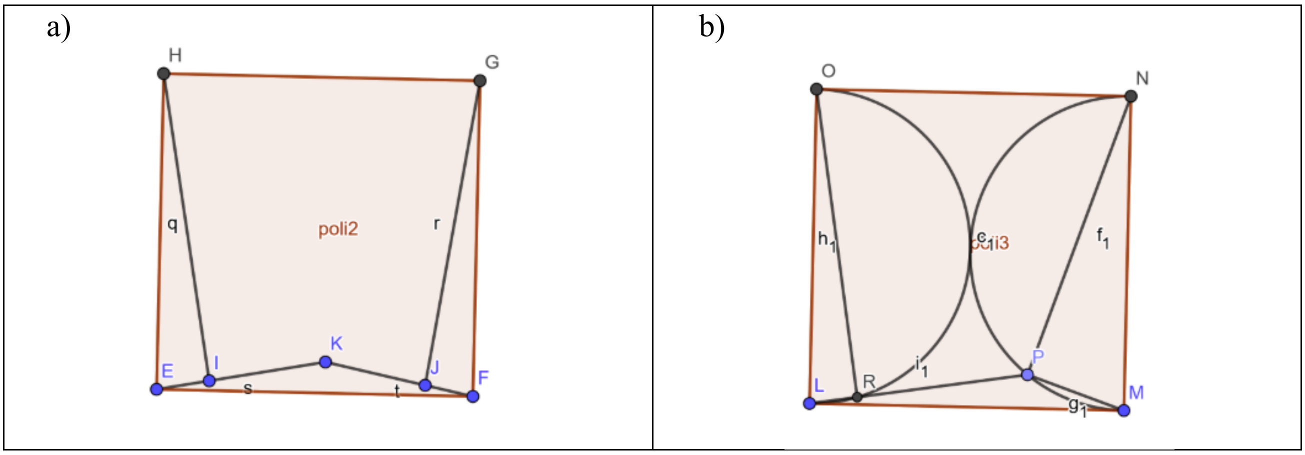

Figure 3.

Analysis of two reports of students who concluded that the solution is (approximately) the square

What about the remaining three groups? As said before, the students in these groups conjectured that the solution for (their interpretation of) the problem was the square itself, but were puzzled by the hint which straightforwardly told them that this was not the case. Of course, the hint was specifically given in order to prompt students to think about other non-obvious solutions and help them in identifying the correct one. However, some of the subjects instead gave an argument for affirming that the solution was the square itself and then, since this had been excluded by default, proceeded to propose an "approximate solution" to the problem. Let us see how by analysing the reports of two representative groups (here called E and F) of this latter portion of students. The remaining Group G behaved had a similar report as to Group E.

Indeed, the observer of Group E wrote that her colleagues considered the diagonals first but then

> After some reflection, they discussed on the fact that by considering the diagonals of the square, in order to visit all the cities, they necessarily needed to use one of the sides. Given the impossibility of doing this, they abandoned this idea.

This gives us a hint on the fact that they were probably not interpreting the problem correctly, but it is difficult to see which interpretation they were considering. Possibly they imposed to themselves the self-limitation of not being able to cross the same path or point twice. In any case, they were convinced that the solution to their interpretation of the problem had to be the square. Since this solution was ruled out by the hint, they then reasoned as follows.

> […] they thought of creating polylines [delle spezzate], not necessarily coinciding with the diagonals, which approximated best the perimeter and so that their point of intersection lied on the square's axis. Initially they considered these just on two sides of the square while on the others they considered the diagonals. In order to understand if the minimal path was that formed by the polylines on two sides and the diagonals on the other two or rather was the path consisting of polylines on all four sides they decided […] to calculate which one was shorter […] Therefore […] their final conjecture was that of choosing, as minimal path, the one consisting of polylines which best approximate the square's perimeter.

What happened here? Using the theoretical framework described above, we may interpret the procedure the students went through as first a failure to see that other solutions are possible (and hence a difficulty in imagination) together with an overgeneralization of the particular case they took into consideration for formulating the conjecture (and hence a failure in intuition). Indeed, they constructed using Geogebra the two configurations displayed in Figure 3.a and 3.b below, then they calculated their respective perimeters (notice that for Figure 3.a this includes the dotted diagonals) and finally conjectured that the solution should be the latter.

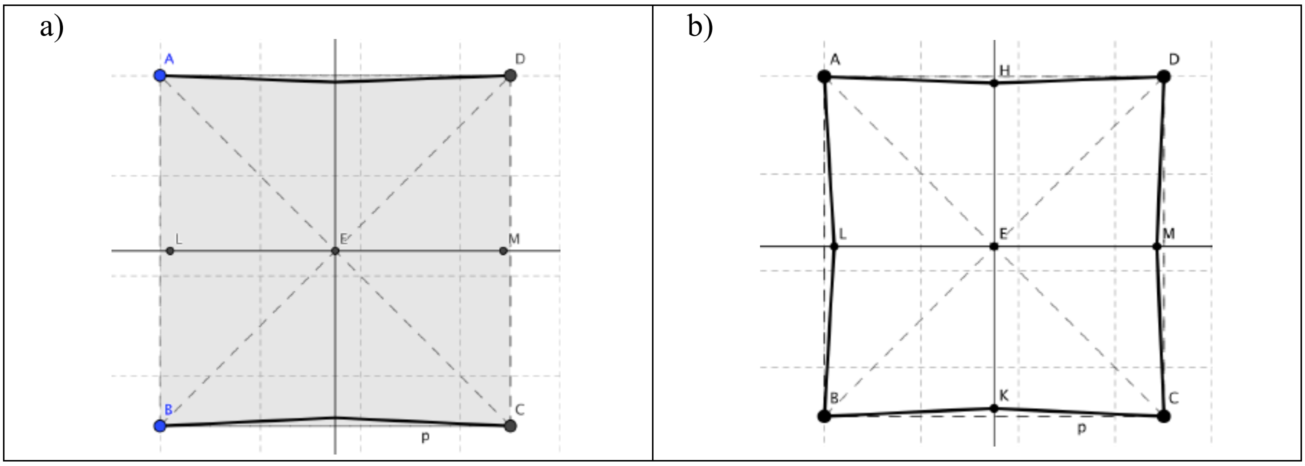

Figure 4.

Furthermore, what is perhaps most striking here is the struggle the students experienced in formulating the consequences of their conjecture. Since the latter points to the fact that the solution should be the square itself, but since also the teacher had ruled out this possibility, these master students in mathematics are then forced to conclude a mathematical impossibility: the minimal path is the one which best approximates the square itself, despite being no such path!

Similarly, Group F started by considering the diagonals of the square but then

> […] came the idea of creating a square in the centre [of the original square] whose side can vary between 0 and l [the side of the square] linking any of its vertex to one and only one city […]

So, they used Geogebra to represent this situation (as displayed in Figure 5 where point R can vary over side QN) and concluded that

> […] point *R* must be as close as possible to point *N* for having the minimal path: this means that the two squares must have roughly the same side […] The minimal path is given by the approximation of the square having as vertices the four cities.

Here again for these students "the" solution is "the minimal path" consisting of the best possible approximation to the square itself, despite this being a mathematical impossibility.

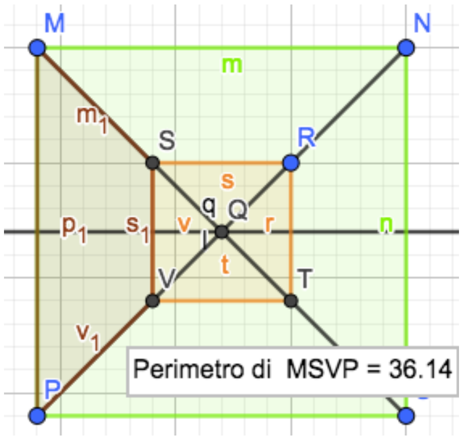

Figure 5.

CONCLUSIONS

It seems reasonable to conclude from this preliminary analysis that overall the main obstacle the subjects experienced was a combination of failure of (active) imagination and failure of intuition fostered by false deduction. First, students' failure in the imaginative step may have been caused by the fact that the square itself prompted to a false solution: in the words of (some of) the subjects, the square "offered" or cued the diagonals as the solution for presumably all Groups A, B, C and D. A similar phenomenon is probably safe to assume having played a role also for Groups E, F and G in convincing them that the square itself (or its "approximation") was the solution. Second, the failure in the intuitive step (shared by all groups) was most likely caused, as already said, by an overgeneralization fostered by false deduction as explicit reference to this phenomenon are present in the reports of both the first and the second portion of students. Indeed, this is immediately clear on the one hand from our analysis above of the quoted passages from Groups A, B, E and F (as well as the not here reported Groups C, D and G) where, invariably, the consideration of particular cases is invariably taken by the subjects as confirmation of the truth of their conjecture in the general case. As a linguistic observation, this overgeneralizing procedure is also reflected by the awkward usage of the term "deduction" in the quoted passage of Group B, where "deduced" is used as a synonym of "conjectured" or simply "thought of".

Of course, we do not mean to understand these preliminary conclusions in purely psychological terms as dependent solely on the students' internal faculties, them being completely separated from the context these students were immersed in. On the contrary, the difficulties we have outlined in this paper render perhaps most evident the fact that the kind of undergraduate mathematical training to which these students were exposed fails (at least in this case) as a training for visual conjecturing problem solving. Indeed, as already said above, these students were trained almost exclusively on traditionally-presented mathematical content focusing on the study of ready-made theorems while very rarely being encouraged to work on their own conjectures. Moreover, as we have seen with Group E, F and G, it persists (at least for these students) a difficulty in reasoning outside of the didactic contract (on this concept see Brousseau, Sarrazy and Novotná 2014) they assume to be in place, together with a difficulty in questioning the authority of the teacher or else a difficulty in believing in their own mathematical reasoning: i.e. they prefer to conclude a mathematical impossibility rather than dismissing their own mathematical conclusion or rather than dismissing what they perceive to be a normative characterization of the problem coming from the teacher. A deeper (and more institutionally-oriented) study into this phenomenon would be required in order to reach more than tentative conclusions on this matter. A result of such study could perhaps lead to suggestions in developing the university mathematics curriculum in favour of a greater exposure of students to visual conjecturing problems. It remains however an open question whether such exposure may in general succeed in the training of imagination and intuition and, in general, whether these faculties are susceptible to be trained at all. This, in turn remains a psycho-pedagogical matter over which further investigation would be needed.

References

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, 52 (3), pp. 215-241.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *Zentralblatt für Didaktik der Mathematik* 34 (3), pp. 66-72.

Brousseau, G., Sarrazy, B., & Novotná, J. (2014). Didactic Contract in Mathematics Education. In S. Lerman (Ed.) *Encyclopedia of Mathematics Education*. Dordrecht: Springer.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach*. Dordecht: Reidel.

Giardino, V. (2010). Intuition and Visualization in Mathematical Problem Solving. *Topoi* 29, pp. 29–39.

Kadunz, G., & Yerushalmy M. (2015) Visualization in the Teaching and Learning of Mathematics. In S. Cho (Ed.) *The Proceedings of the 12th International Congress on Mathematical Education*. Clam: Springer.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education*, Rotterdam: Sense Publishers, pp. 205-235.

Presmeg, N. C. (2014). Visualization and Learning in Mathematics Education. In S. Lerman (Ed.) *Encyclopedia of Mathematics Education*. Dordrecht: Springer.