INTUITION AND HIGHER MATHEMATICAL COGNITION

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Turing's distinction

> Mathematical reasoning [when considered in the restrictive sense of determining the truth or falsity of propositions] may be regarded rather schematically as the exercise of a combination of two faculties, which we may call **intuition** and **ingenuity**. The activity of the intuition consists in **making spontaneous judgments which are not the result of conscious trains of reasoning** [...] The exercise of ingenuity in mathematics consists in **aiding the intuition through suitable arrangements of propositions, and perhaps geometrical figures or drawings**. It is intended that when these are really well arranged the validity of the intuitive steps which are required cannot seriously be doubted. **The parts played by these two faculties differ of course from occasion to occasion, and from mathematician to mathematician. This arbitrariness can be removed by the introduction of a formal logic.** The necessity for using the intuition is then greatly reduced by setting down formal rules for carrying out inferences which are always intuitively valid. **When working with a formal logic, the idea of ingenuity takes a more definite shape.** In general a formal logic, will be framed so as to admit a considerable variety of possible steps in any stage in a proof. **Ingenuity will then determine which steps are the more profitable for the purpose of proving a particular proposition.** (Turing 1939)

First analogy: Dual-process accounts of human reasoning distinguish between two kinds of general thinking: one **unconscious, fast and intuitive which has similarities with perception**, the other **conscious, slow and reflective superseding hypothetical thinking and deduction** (Evans and Stanovich 2013)

An informal example

Statement: If a number is smaller than another number, then the square of the former is smaller than the square of the latter:

$$\forall a, b \in \mathbb{N}(a < b \to a^2 < b^2)$$

Proof: In ordinary mathematical language, we could say that a proof of this statement is elementarily obtained by the application of **the (axiomatic) definition of "being smaller than":** a is smaller than b means that there is some number $c > 0$ such that $a + c = b$.

Therefore, we can write $(a + c)^2 = b^2$ and hence

$$a^2 + c^2 + 2ac = b^2. \; *$$

Since we are assuming $c > 0$ it is easy to see that $c^2 + 2ac > 0$. **

But this in turn means that there exists a positive number $k = c^2 + 2ac$ such that $a^2 + k = b^2$ and thus we have found that $a^2 < b^2$, again **by the definition of "being smaller than".**

More formally, the statement above can be given **a pedantic step-by-step proof** by application **elementary rules of logic whose validity is almost impossible to question.**

$$(1)\ \frac{\forall x P(x)}{P(\alpha)}\ \text{, for any } \alpha \qquad (2)\ \frac{\exists x P(x)}{P(\alpha)}\ \text{, for some } \alpha$$

$$(3)\ \frac{P(x)}{\forall x P(x)}\ \text{, for } x \text{ free in } P \qquad (4)\ \frac{P(\alpha)}{\exists x P(x)}\ \text{, for } x \text{ a fresh variable}$$

$$(5)\ \frac{P \to \exists x Q(x)}{\exists x(P \to Q(x))} \qquad (6)\ \frac{\exists x(P \to Q(x))}{P \to \exists x Q(x)} \qquad (7)\ \frac{P \leftrightarrow Q}{P \to Q} \qquad (8)\ \frac{P \leftrightarrow Q}{Q \leftrightarrow P}$$

$$(9)\ \frac{P \to Q \wedge R}{P \to R \wedge Q} \qquad (10)\ \frac{P \to Q \wedge R \qquad Q \to S}{P \to S \wedge R}\ \text{, possibly } R = \emptyset$$

AXIOM

$$(1)\ \frac{\forall x,y(x<y \leftrightarrow \exists z(z>0 \wedge x+z=y))}{\forall y(a<y \leftrightarrow \exists z(z>0 \wedge a+z=y))}$$
$$(1)\ \frac{\forall y(a<y \leftrightarrow \exists z(z>0 \wedge a+z=y))}{a<b \leftrightarrow \exists z(z>0 \wedge a+z=b)}$$
$$(10)\ \frac{a<b \leftrightarrow \exists z(z>0 \wedge a+z=b)}{a<b \to \exists z(z>0 \wedge a+z=b)}$$
$$(5)\ \frac{a<b \to \exists z(z>0 \wedge a+z=b)}{\exists z(a<b \to (z>0 \wedge a+z=b))}$$
$$(2)\ \frac{\exists z(a<b \to (z>0 \wedge a+z=b))}{a<b \to c>0 \wedge a+c=b}$$
$$(9)\ \frac{a<b \to c>0 \wedge a+c=b}{a<b \to a+c=b \wedge c>0}$$

THEOREM 1 *

$$(1)\ \frac{\forall x,y,z(x+y=z \to x^2+y^2+2xy=z^2)}{\forall y,z(a+y=z \to a^2+y^2+2ay=z^2)}$$
$$(1)\ \frac{\forall y,z(a+y=z \to a^2+y^2+2ay=z^2)}{\forall z(a+c=z \to a^2+c^2+2ac=z^2)}$$
$$(1)\ \frac{\forall z(a+c=z \to a^2+c^2+2ac=z^2)}{a+c=b \to a^2+c^2+2ac=b^2}$$

$$(10)\ \frac{a<b \to a+c=b \wedge c>0 \qquad a+c=b \to a^2+c^2+2ac=b^2}{a<b \to a^2+c^2+2ac=b^2 \wedge c>0}$$
$$(9)\ \frac{a<b \to a^2+c^2+2ac=b^2 \wedge c>0}{a<b \to c>0 \wedge a^2+c^2+2ac=b^2}$$

THEOREM 2 **

$$(1)\ \frac{\forall x,y(x>0 \to x^2+y>0)}{\forall y(c>0 \to c^2+y>0)}$$
$$(1)\ \frac{\forall y(c>0 \to c^2+y>0)}{c>0 \to c^2+2ac>0}$$

$$(10)\ \frac{a<b \to c>0 \wedge a^2+c^2+2ac=b^2 \qquad c>0 \to c^2+2ac>0}{a<b \to (c^2+2ac>0 \wedge a^2+c^2+2ac=b^2)}$$
$$(4)\ \frac{a<b \to (c^2+2ac>0 \wedge a^2+c^2+2ac=b^2)}{\exists z(a<b \to (z>0 \wedge a^2+z=b^2))}$$
$$(6)\ \frac{\exists z(a<b \to (z>0 \wedge a^2+z=b^2))}{a<b \to \exists z(z>0 \wedge a^2+z=b^2)}$$

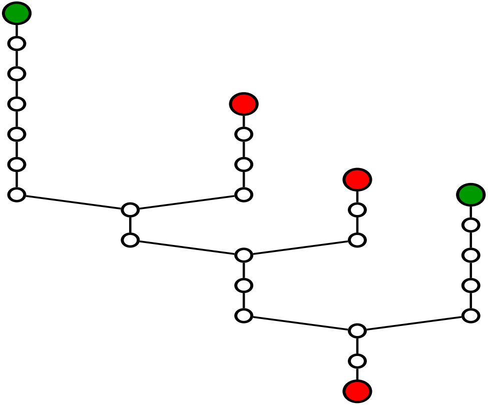
AXIOM

$$(1)\ \frac{\forall x,y(x<y \leftrightarrow \exists z(z>0 \wedge x+z=y))}{\forall y(a^2<y \leftrightarrow \exists z(z>0 \wedge a^2+z=y))}$$
$$(1)\ \frac{\forall y(a^2<y \leftrightarrow \exists z(z>0 \wedge a^2+z=y))}{a^2<b^2 \leftrightarrow \exists z(z>0 \wedge a^2+z=b^2)}$$
$$(8)\ \frac{a^2<b^2 \leftrightarrow \exists z(z>0 \wedge a^2+z=b^2)}{\exists z(z>0 \wedge a^2+z=b^2) \leftrightarrow a^2<b^2}$$
$$(7)\ \frac{\exists z(z>0 \wedge a^2+z=b^2) \leftrightarrow a^2<b^2}{\exists z(z>0 \wedge a^2+z=b^2) \to a^2<b^2}$$

$$(10)\ \frac{a<b \to \exists z(z>0 \wedge a^2+z=b^2) \qquad \exists z(z>0 \wedge a^2+z=b^2) \to a^2<b^2}{a<b \to a^2<b^2}$$
$$(3)\ \frac{a<b \to a^2<b^2}{\forall b(a<b \to a^2<b^2)}$$
$$(3)\ \frac{\forall b(a<b \to a^2<b^2)}{\forall a,b(a<b \to a^2<b^2)}$$

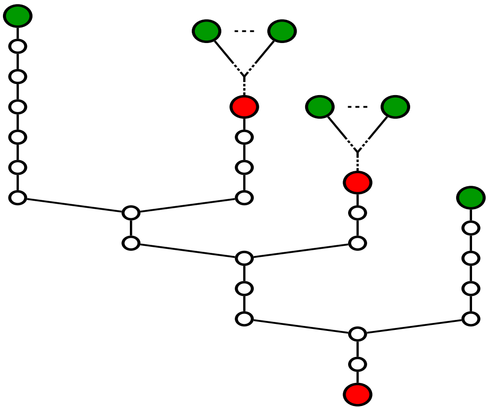
CONCLUSION

Mechanizing the process 1



In general, the proof of any mathematical statement can be arranged as a formal proof, thus taking a structure similar to that of a tree, where the leaves are the premises of the argument and the root is the conclusion of the argument. Starting from the bottom of the tree, the first node (the root of the tree) contains the sentence we want to prove, and each line above it follows directly from one or more premises by application of a rule of logic. At the very top of the tree (at the leaves) there are those sentences which can be taken to be true without proof: either axioms or previously established theorem. In this way the appeal to intuition is confined to the strictly necessary (axioms and rules of logic) and ingenuity takes a very precise form.

Mechanizing the process 2



Notice that any previously derived theorem, by definition, can in turn be the root of another tree. With plenty of patience we can reduce to the situation in which our tree has only axioms as leaves.

Now, since each rule of logic can use only a finite number of premises, a machine (or a human with plenty of time and patience), can just enumerate all the possible premises of a statement and, starting from the bottom, build all the possible trees having such statement as root. It follows that we can program a machine which subsequently builds all the possible trees having as root the chosen statement such that each node is obtained by application of one of the rules. If we ask it to stop whenever it finds a tree having only axioms as leaves, then we have a proof-finding machine!

Gödelian objections

> We are always able to obtain from the rules of a formal logic a method of enumerating the propositions proved by its means. We then imagine that all proofs take the form of a search through this enumeration for the theorem for which a proof is desired. In this way ingenuity is replaced by patience. (Turing 1939)

It was believed in the past that it would eventually be feasible to indefinitely extend the mechanistic endeavour to the point of programming a machine capable of completely replacing humans in doing mathematics (this was roughly Hilbert's program).

> In pre-Gödel times it was thought by some that it would probably be possible to carry this programme to such a point that all the intuitive judgments of mathematics could be replaced by a finite number of these rules. The necessity for intuition would then be entirely eliminated. In our discussions, however, we have gone to the opposite extreme and eliminated not intuition but ingenuity, and this in spite of the fact that our aim has been much the same direction. (Turing 1939)

What stems from the incompleteness theorems is that

1) there is no enumeration of all the possible axioms of mathematics that can be recognised as such;
2) furthermore, the Turing theorem (itself a version of the incompleteness theorem for theoretical computer science) ensures that it is not possible to build a machine that decides in advance whether there is or there is not a proof of a given statement (for all statements).

The need for intuition

As a consequence, even if it turns out to be possible to replace ingenuity with good mechanical programming, it is not in practice possible to circumvent the role of intuition. Indeed, intuition remains a fundamental component in mathematical practice for

1) perceiving the truth of axioms and the correctness of definitions;
2) perceiving that a proof of a statement exists if it does or does not exists if it does not.

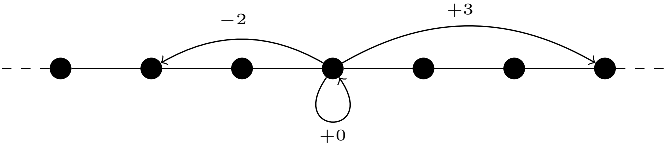
In our example,

1) Even if we can program a machine that, say, finds a proof-tree starting from the axiom that expresses the meaning of "being smaller than" down to $a < b \to a^2 < b^2$, the machine is not understanding what it really means "to be smaller than". The machine was just hard-wired the axiom $\forall x, y(x < y \to x^2 < y^2)$ as a legal move in this abstract proof-tree-building game. On the contrary, most mathematicians introspectively perceive the meaning of "being smaller than" based on the mental representation of numbers as being on an imaginary line one after another.
2) We knew in advance that there was a proof of $a < b \to a^2 < b^2$. It follows that a suitably programmed machine would eventually find it for good. However, if we are presented with another statement (of which we do not know whether there is a proof or not), then a similarly programmed machine would eventually find a proof of the statement only if there is indeed a proof of the statement. On the contrary, if no such proof exists, the machine will go on building tree after tree indefinitely, never returning an answer.

The simplest spatial model for arithmetical intuition

One could easily argue that the truth of our definition of “being smaller than”, $a < b \to \exists z(a + z = b)$, ultimately rests on the correctness of our introspective representation of numbers as resting on a line separated by magnitudes (or distances) which themselves can be made to correspond to numbers on the line. In such representation

- “being smaller than a” means “being on the left with respect to a”;
- 0 is the unique magnitude that if added to another quantity corresponds to not moving on the line;
- “adding c to a” thus means moving from a to the right of distance c, while “subtracting c from a” means moving from a to the left of distance c.



Second analogy: According to the mental number line account of number representation, numbers are represented by the mind on a horizontal line in ascending order from left to right or from right to left depending on the subjects’ writing direction (Dehaene et al. 1993). Furthermore, this faculty of representation would be innate and linked to specific areas of the brain connected to the processing of spatial coordinates (Umiltà, Priftis and Zorzi, 2009).

Turing's proposal

However, how is it possible to mechanize (and thus understand precisely) the general concept of intuition? Let us examine briefly the mature Turing's proposal:

> In the process of trying to imitate an adult human mind we are bound to think a good deal about the process which has brought it to the state that it is in [...] Instead of trying to simulate an adult mind, why not rather try to produce one which simulates the child's? If this [machine] were then subjected to an appropriate course of education one would obtain the adult brain. [...] We have thus divided our problem into two parts. The child-programme and the education process. [...] It can also be maintained that it is best to provide the machine with the best sense organs that money can buy, and then teach it to understand and speak English. This process could follow the normal teaching of a child. Things would be pointed out and named, etc. (Turing 1950)

Turing, very ahead of his time, was here promoting the development of (embodied) cognitive models of the human mind: since – as we have seen – there is no way to circumvent the need for intuition by other means, the obvious step for imitating the human mind is trying to give a machine "the best sense organs that money can buy": make it into a closer model of the body!

Third analogy: Cognitive embodied accounts of knowledge as emerging in connection with our perceptual and motor systems including (but not limited to) perception of movement and body shapes. In particular, mathematics is a mental creation that evolved from our manyfold practice with objects of the world (Lakoff and Nuñez 2000).

The value of introspection

As we have seen three different modes of reasoning on mathematical practice lead to very similar conclusions about intuition:

- Introspection of one's own mathematical practice (Hadamard, Poincaré, Turing).
- Empirical results about other people's mathematical practice (Dehaene, Nuñez, Lakoff).
- Theoretical results on mathematical practice in general (Gödel, Turing).

Introspection ← **Empirical results** → **Theory**

Subjective ... Objective

Introspective accounts of mathematical practice – especially when congruent with theoretical findings – should be valued more when dealing with higher mathematics, for in this context empirical research may not be feasible.

- Very few people have a sufficient understanding of higher mathematics.
- Even fewer have a sufficient understanding of higher mathematics AND are willing and sufficiently skilled in empirical psychology to design and perform tests on others.

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